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The Binomial Option Pricing Model

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January 2002

Consider a non-dividend paying stock whose price is initially S_0. Divide time into small time intervals of length Δt. A time interval will be referred to as a period. Denote by S the initial stock price at the beginning of a time interval. Assume that in each time interval the stock price moves either to uS (an "up" movement) or to dS (a "down" movement). The parameters u and d are equal to one plus the realized return during the time interval. In general $u>1$ and $d<1$. Let q be the probability of an "up" movement and $(1-q)$ be the probability of a "down" movement.
Let r_f be the risk-free interest rate for each time interval. To avoid arbitrage opportunities, one should assume that $u>1+r_f>d$.

Valuing a one-period call option

Consider a call option with a strike price X and one-period to maturity. Let C_0 be its current value and C_1 be its value in one period.
As the maturity of the call option is one period, C_1 is the terminal value of the option. It can take two possible values:

$$C_u = \text{Max}(0, uS_0 - X) \qquad \text{if } S_1 = uS_0 \qquad [1.a]$$

$$C_d = \text{Max}(0, dS_0 - X) \qquad \text{if } S_1 = dS_0 \qquad [1.b]$$

Note that the reason for assuming the option's maturity is one period is that it allows knowing the possible values of the option at the end of the period.
The question is how to find C_0?
The standard answer would be to discount the expected future value of the option using a risk-adjusted discount rate. Unfortunately, this approach leads to a dead end. The key insight of Black, Scholes and Merton is to have turned around the problem by showing that it is possible to create a riskless portfolio by combining the underlying stock and the option. As the portfolio is riskless, its value can be obtained by discounting its future value (which is known) at the risk-free interest rate.
To see this, consider a portfolio composed of a long position on δ shares of the stock (δ is the Greek letter delta) and a short position on one call option. Stated differently, the portfolio is created by buying δ shares and selling one call option. The initial value of this portfolio V_0 is:

$$V_0 = \delta S_0 - C_0$$

In this equation, a minus sign indicates a short position whereas a plus sign indicates a long position.
In order for this portfolio to be riskless, its future value should be the same whatever the future stock price:

$$\delta u S_0 - C_u = \delta d S_0 - C_d$$

Solving for δ gives the number of shares to buy:

$$\delta = (C_u - C_d)/(uS_0 - dS_0) \qquad [2]$$

Note that, as $C_u > C_d$, $\delta > 0$.
Let us check that we achieve the desired result with this value of δ.
Assume first $dS_0 < X < uS_0$ (this assumption ensures that the decision to exercise or not the option is not known at the beginning of the period). Then:

$C_u = uS_0 - X$ (the option ends up being exercised)

$C_d = 0$ (the option ends up being worthless)

$$\delta = [(uS_0 - X)/ (uS_0 - dS_0)]$$

Consequently, the possible values of the portfolio at the end of the period are:

$$V_u = [(uS_0 - X)/ (uS_0 - dS_0)]uS_0 - (uS_0 - X) = [(uS_0 - X)d]/(u-d)$$

$$V_d = [(uS_0 - X)/ (uS_0 - dS_0)]dS_0 - \quad 0 \qquad = [(uS_0 - X)d]/(u-d)$$

The portfolio is indeed riskless.

Two other situations might occur:

$$dS_0 < uS_0 < X$$

This case is trivial. The option will never be exercized. Whatever the evolution of the stock price, it will end up out of the money. The option is worthless today.

$$X < dS_0 < uS_0$$

In this case, the option will be exercized in both the "up" and "down" cases. Buying the call option is identical to buying the stock forward for a delivery price X.

$$C_u = uS_0 - X$$

$$C_d = dS_0 - X$$

$$\delta = [(uS_0 - dS_0)/ (uS_0 - dS_0)] = 1$$

Consequently, the possible values of the portfolio at the end of the period are:

$$V_u = uS_0 - (uS_0 - X) = X$$

$$V_d = dS_0 - (dS_0 - X) = X$$

The portfolio is indeed riskless.

Once the receipe to create a riskless portfolio is known(in this simple setting, it only involves knowing how many shares to buy), valuing this portfolio is straightforward. As the future value is known, its present value is obtained by discounting the future value at the risk-free interest rate.
Denoting V_1 for the future value, the current value of the riskless portfolio is:

$$V_0 = V_1/(1+r_f)$$

As:

$$V_0 = \delta S_0 - C_0$$

we get:

$$\delta S_0 - C_0 = V_1/(1+r_f)$$

Solving for C_0 leads to:

$$C_0 = \delta S_0 - V_1/(1+r_f) \qquad [3]$$

The interpretation is the following. Buying a call option is equivalent to buying δ shares and borrowing. This formula is a preview of the Black-Scholes formula for European call options.
We get go one step further. First note that:

$$V_1 = \delta u S_0 - C_u = \delta d S_0 - C_d$$

Now replace δ and V_1 by their respective values in [3]. This leads to a closed form solution for C_0:

$$C_0 = [p.C_u + (1-p).C_d]/(1+r_f) \qquad [4]$$

with:

$$p = (1+r_f - d)/(u-d) \text{ and } (1-p) = (u-1-r_f)/(u-d) \qquad [5]$$

As $d < 1+r_f < u$, p and $(1-p)$ are numbers between 0 and 1. They look like probabilities.
Let suppose that they represents some sort of probabilities associated with up and down movements. The interpretation of [4] would then pretty straightforward. The numerator would be the expected value of the option in one period. The current value is obtained by discounting the expected future value at the risk-free interest rate.
The puzzling thing is that p is different from the probability q assumed initially for an up movement. As a matter of fact, this probability q does not play any role for valuing the option. What happened?
In fact, p looks like a probability but is not the true probability of an up movement (the true probability is q). p is a "pseudo probability". It is the probability of a an movement *assuming that the expected return on the stock is equation to the risk-free interest rate*.
If the probability of an up movement was p, the expected return from the stock would be:

$$p(u-1) + (1-p)(d-1) = p(u-d)+d-1$$

Now, set this expected return equal to the risk-free interest rate:

$$p(u-d) + d - 1 = r_f$$

This will hold if:

$$p = (1+r_f - d)/(u-d)$$

This is precisely the value found for p in [5].
So p has a nice interpretation. It is the probability of an up movement in a *risk-neutral world*. In such a world, investors are indifferent to risk. They require the same expected return on all securities: the risk-free interest rate.
What equation [4] and [5] tell us is that, to value the option, one should leave the real world and move to a virtual world were everyone is risk-neutral. The move from the real world to the risk-neutral world is achieved by changing the probabilities of the up and down movement so as to have an expected return on all securities (including the underlying asset) equal to the risk-free interest

rate. Prices are identical in the real world and in the risk-neutral world. Probabilities associated with price movements are different.

Example 1: valuing a one-period call option

Suppose that we want to value a one-month call option on a non dividend paying stock. The current price of the stock is €100 and the strike price is €100. The risk-free interest rate with continuous compounding is 0.5% per month. It is known that the stock price at the end of one month will be either €110 (with probability 0.6) or €90 (with probability 0.4).

Let us value this option using a one-period binomial with a time step equal to one month (Δt=1/12). The variable used have the following values:

$S_0 = 100 \quad X = 100$

$u = 1.1 \quad d = 0.9 \quad r_f = 0.5\%$

$q = 0.6$

The expected return on the stock is:

Expected return = (0.6)(+10%) + (0.4)(-10%) = +2%

The possible future values of the call options are:

$C_u = \text{Max}(0, 110\text{-}100) = 10$

$C_d = \text{Max}(0, 90\text{ -}100) = 0$

The probability p of a up movement in a risk neutral world is:

$p = (1.005 - 0.9)/(1.1 - 0.9) = 0.525$

The number of shares to buy in conjunction with the sale of one call option to create a riskless portfolio is:

$\delta = (10\text{-}0)/(110\text{-}90) = 0.5$

We can check that buying 0.5 shares and selling a call leads to a riskless portfolio.

The future value (in one period) of this portfolio is:

0.5 x 110 - 10 = 45 if the stock price is 110 in period 1;

0.5 x 90 - 0 = 45 if the stock price is 90 in period 1..

The future value of this portfolio is $V_1 = 45$. Its current value is:

$V_0 = V_1/(1+r_f) = 45/1.005 = 44.78$

The value of the call option is:

$C_0 = 0.5 \text{ x } 100 – 44.78 = €5.22$

This value can be obtained directly using the one-period option valuation formula [4]:

$C_0 = (0.525 \text{ x } 10)/1.005 = €5.22$

Valuing a put option.

The preceding reasoning for valuing a call option can be easily transposed to a value one-period put option. Let P_0 be its current value and P_1 be its value in one period.

As the maturity of the call option is one period, P_1 is the terminal value of the option. It can take two possible values:

$$P_u = \text{Max}\ (0, X - uS_0) \qquad \text{if } S_1 = uS_0 \qquad [6.a]$$

$$P_d = \text{Max}\ (0, X - dS_0) \qquad \text{if } S_1 = dS_0 \qquad [6.b]$$

Note that $P_u < P_d$ (a put option becomes valuable is the stock price decreases).
Rather than reproducing the same logic, we will present the valuation in a slightly different way. Go back to equation [3] for a call option. This formula tells us how to create a synthetic call option. A synthetic call option is a portfolio that behaves like a call option. Its value at the end of the period is equal to the value of the call option. Formula [3] tells us that this portfolio will consist of δ shares combined to a position on a zero-coupon. A synthetic call option consist on a long position on the stock ($\delta>0$) and a short position on a zero-coupon.
Let's see how a create a synthetic put option. The portfolio consist of δ shares and an amount M in a zero-coupon. Keep in mind that we should buy shares if δ is positive, we would be short on the stock if δ is negative. Similarly, a positive value for M would mean that we invest (we buy a zero-coupon), a negative value for M would mean that we borrow (a short position on a zero-coupon).
To reproduce the future value of the put option, δ and M should satisfy the two following conditions:

$$\delta uS_0 + M(1+r_f) = P_u \qquad [7.a]$$

$$\delta dS_0 + M(1+r_f) = P_d \qquad [7.b]$$

Subtract [7.b] from [7.a] and solve for δ

$$\delta = (P_u - P_d)/(uS_0 - dS_0) \qquad [8]$$

Compare [8] with [2]. Except for changes of notations, the two formulas are identical. Notice however that, as $P_u<P_d$, δ is negative.
Now replace δ by its value [8] in [7.a] and solve for M:

$$M = (P_d u - P_u d)/[(u-d)(1+r_f)] \qquad [9]$$

Is M positive or negative?
If $dS_0<X<uS_0$: $P_d = X-dS_0 >0$ and $P_u =0 \Rightarrow M>0$
If $dS_0<uS_0<X$: $P_d = X-dS_0 >0$ and $P_u = X-uS_0 \Rightarrow M=X/(1+r_f) >0$
We have so found that the replicating portfolio for the one-period put option is composed of a short position on $|\delta|$ shares (δ being a negative number, the number of shares to short is equal to its absolute value) and an investment M in the riskless asset. The value of the put option is equal to the value of the replicating portfolio:

$$P_0 = \delta S_0 + M \qquad [10]$$

The same result could have been obtained by creating a riskless portfolio as for the call option. The composition of this portfolio can be deducted from [10] by writing the equality as:

$$M = -\delta S_0 + P_0$$

The left-hand side of the equality is the amount to invested the riskless asset. The right-hand side provides an alternative strategy to achieve the same objective. It consists in buying $|\delta|$ shares (remember that δ is negative for a put option) and buying one put option.
The formula for P_0 is obtained by replacing δ and M in [10] by their respective values [8] and [9] and solving for P_0. It leads to:

$$P_0 = [p.P_u + (1-p).P_d]/(1+r_f) \qquad [11]$$

with:

$$p = (1+r_f - d)/(u-d) \text{ and } (1-p) = (u-1-r_f)/(u-d) \qquad [12]$$

Except for obvious changes of notations, these are the same expression as equation [4] and [5] derived for the one-period call option.

Example 2: valuing a one-period put option

Suppose that we want to value a one-month put option on a non dividend paying stock. The current price is of the stock is €100 and the strike price is €100. The risk-free interest rate with continuous compounding is 0.5% per month. It is known that the stock price at the end of one month will be either €110 (with probability 0.6) or €90 (with probability 0.4).
Let us value this option using a one-period binomial with a time step equal to one month (Δt=1/12).
The variable used have the following values:

$S_0 = 100 \quad X = 100$
$u = 1.1 \quad d = 0.9 \quad r_f = 0.5\%$
$q = 0.6$

The expected return on the stock is:

Expected return = (0.6)(+10%) + (0.4)(-10%) = +2%

The possible future values of the put option are:

$P_u = \text{Max}(0, 100\text{-}110) = 0$

$P_d = \text{Max}(0, 100\ \text{-}90) = 10$

The probability p of an up movement in a risk neutral world is:

$p = (1.005 - 0.9)/(1.1 - 0.9)=0.525$

The delta of the option gives the number of share in the replicating portfolio:

$\delta = (0\text{-}10)/(110\text{-}90) = -0.5$

The amount to invest is:

$M = [(10)(1.10) – (0)(0.9)]/[(1.10\text{-}0.9)(1.005)]= 54.73$

We can check that being short on 0.5 shares and investing 54.73 at 0.5% replicates the future value of the put option.
The future value (in one period) of this portfolio is:

-0.5 x 110 + 54.73 x 1.005 = 0 if the stock price is 110 in period 1;

-0.5 x 90 + 54.73 x 1.005 = 15 if the stock price is 90 in period 1.

The value of the put option is:

C_0 = -0.5 x 100 + 54.73 = €4.73

This value can be obtained directly using the one-period option valuation formula [11]:

C_0 = [0.525 x 0 + (1-0.525) x 10)/1.005 = €4.73

The general one-period binomial option formula

The previous analysis leads us to very general procedure to value any derivative security at the beginning of a time interval. Let S be the stock price and V be the value a derivative security at the beginning of any period. The stock price moves either to uS or to dS. The value of the derivative security at the end of the period depend on the evolution of the stock. Suppose that the two possible values V_u and V_d are known. Then, based on the previous analysis:

Step 1: calculate the risk-neutral probabilities

$$p = (1+r_f - d)/(u-d) \text{ and } (1-p) = (u-1-r_f)/(u-d) \qquad [12]$$

Step 2: value the option by discounting the expected future value in a risk-neutral world at the risk-free interest rate

$$V = [p.V_u + (1-p).V_d]/(1+r_f) \qquad [13]$$

The value of the option is also equal to the value of the replicating portfolio

$$V=\delta S + M \qquad [14]$$

The delta of the option is:

$$\delta = (V_u - V_d)/(uS - dS) \qquad [15]$$

The amount invested at the risk-free interest rate in the replicating portfolio (a negative value means borrowing) is

$$M = (V_d u - V_u d)/[(u-d)(1+r_f)] \qquad [16]$$

Example 3: valuing a binary option

A *cash-or-nothing call* option is a contract which pays a fixed amount if the stock price ends up above the strike price and nothing is the stock price ends up below the strike price.
Suppose that we want to value a one-month cash-or-nothing call option on a non dividend paying stock. The current price is of the stock is €100 and the strike price is €100. The option pays €50 if the stock price in one month is above the strike price. The risk-free interest rate is 0.5% per month. It is known that the stock price at the end of one month will be either €110 (with probability 0.6) or €90 (with probability 0.4).
Let us value this option using a one-period binomial with a time step equal to one month (Δt=1/12). The variable used have the following values:

$S_0 = 100 \quad X = 100$
$u = 1.1 \quad d = 0.9 \quad r_f = 0.5\%$
$q = 0.6$

The possible future values of the option are:

$V_u = 50$

$V_d = 0$

The probability p of a up movement in a risk neutral world is:

$p = (1.005 - 0.9)/(1.1 - 0.9)=0.525$

The delta of the option gives the number of share in the replicating portfolio:

$\delta = (50\text{-}0)/(110\text{-}90) = 2.5$

The amount to invest is:

$M = [(0)(1.10) - (50)(0.9)]/[(1.10\text{-}0.9)(1.005)]= -223.88$

We can check that being long on 2.5 shares and borrowing 223.88 at 0.5% replicates the future value of the option.
The future value (in one period) of this portfolio is:

2.5 x 110 – 223.88 x 1.005 = 50 if the stock price is 110 in period 1;

2.5 x 90 - 223.88 x 1.005 = 0 if the stock price is 90 in period 1.

The value of the cash-or-nothing call option is:

$C_0 = 2.5 \text{ x } 100 - 223.88 =$ €26.12

This value can be obtained directly using the one-period option valuation formula [11]:

$C_0 = [0.525 \text{ x } 50 + (1\text{-}0.525) \text{ x } 0)/1.005 =$ €26.12

Multi-period model

The one-period model is the building block of multi-period valuation. Suppose, for instance, that we wish to value a 3-period call option.
Assume that u and d remain constant through time. The evolution be can represented by a binomial tree as illustrated by figure 1.

Figure 1

Period	0	1	2	3
				u^3S_0
			u^2S_0	
		uS_0		ud^2S_0
	S_0		udS_0	
		dS_0		ud^2S_0
			d^2S_0	
				d^3S_0

To value the option, start from the final maturity. For a call option, the value at maturity is

$C_3 = \text{Max}(0, S_3 - X)$. Then work back through the tree from the end to the beginning. For European option, exercise is only possible at maturiy. As a consequence, the value of the option at each node is obtained using the one-period valuation formula [13]. For an American option, exercise can take place at any time until maturity. One should thus check whether or not to exercise the option before the final maturity. Early exercise will take place for an American whenever the intrinsic value (ie the value from early exercise) is greater than the value of given by [13].
Remember that, if S is the current stock price, the intrinsic value is
$\text{Max}(0, S - X)$ for a call option;
$\text{Max}(0, X\text{-}S)$ for a put option.
As an example, the formula to use at a node for an American put option is:

$$V = \text{Max}\{\text{Max}(0, X\text{-}S)\ ,\ [pV_u\text{+}1\text{-}p)V_d]/(1\text{+}r_f) \qquad [17]$$

Example 4: valuing a three-period European call option

Suppose that we want to value a three-month call option on a non dividend paying stock. The current price is of the stock is €100 and the strike price is €100. The risk-free interest rate with continuous compounding is 0.5% per month. Each month, the stock may go up by 10% (with probability 0.6) or down by 10% (with probability 0.4).
Let us value this option using a binomial tree with a time step equal to one month (Δt=1/12). The variable used have the following values:

$S_0 = 100 \quad X = 100$
$u = 1.1 \quad d = 0.9 \quad r_f = 0.5\%$

The binomial tree for the stock price is:

Period	0	1	2	3
				133.10
			121.00	
		110.00		108.90
	100.00		99.00	
		90.00		89.10
			81.00	
				72.90

The probability p of a up movement in a risk neutral world is:

$$p = (1.005 - 0.9)/(1.1 - 0.9)=0.525$$

The binomial tree for the 3-month call option is:

Period	0	1	2	3
				33.10
			21.50	
		13.43		8.90
	8.16		4.65	
		2.43		0.00
			0.00	
				0

Consider, for instance, the value of the call option in period 2 when the stock price is €99. The option has one-period to maturity and the possible future values are:
C_u = 8.90 if the stock price goes up
C_d = 0 if the stock price goes down.
Using the one-period valuation formula [13], the value of the option at that node is:

$$C = [0.525 \text{ x } 8.90 + (1-0.525) \text{ x } 0]/1.005 = €4.65$$

Example 5: valuing a 3-period American put option

Suppose that we want to value a three-month American put option on the stock of example 3. The strike price is €100.
As a reference, the following table illustrates first the value of a European put option.

Period	0	1	2	3
				0.00
			0.00	
		2.43		0.00
	6.68		5.15	
		11.44		10.90
			18.50	
				27.1

The following tree provides the value for an American option.

Period	0	1	2	3
				0.00
			0.00	
		2.43		0.00
	6.79		5.15	
		11.67		10.90
			19.00	
				27.1

The value of the American option (€6.79) is greater than the value of its European cousin (€6.68). The explanation for this difference is found by looking at the decision on whether to exercise the option in period 2 when the stock price is €81.
The binomial tree for the European option reveals that, if not exercised, the option is worth €18.50. On the other hand, the value of the option is exercised immediately is €19. The decision should be to exercise the option at that node. The initial value of the option captures the optimal early exercise decision.

Dynamic hedging

The composition of the replicating portfolio has been identified in the one-period model. It is interesting to analyze the dynamic of the replicating portfolio in a multi-period setting. It provides insight on dynamic hedging. It also helps to understand how expected return and risk vary for an option.
We defined a replicating portfolio as a combination of the underlying stock and the riskless asset whose evolution during one period is identical to the evolution of the option. The replicating portfolio consists of δ shares of the underlying stock and an investment M in the riskless asset. The value of the option is equal to the value of this replicating portfolio.

$$V = \delta S_0 + M$$

In the binomial tree, the values for δ and M vary at each node. As a consequence, the composition of the replicating portfolio changes over time.
This implies that someone using the replicating portfolio to hedge, for instance, a short position on an option would have to rebalance her portfolio at each node. This strategy is called *dynamic hedging.*

Example 6: hedging a short position on a European call option

Suppose that a financial institution (FI) has sold a three-month call option on a non dividend paying stock. The current price of the stock is €100 and the strike price is €100. The risk-free interest rate is 0.5% per month. Each month, the stock may go up by 10% or down by 10%. The premium received by FI for this option is €10.
The value for this call option has been calculated in example 3 as being equal to €8.61. This is the cost for the financial institution to buy the option. So, the profit on this deal is the difference between the premium received and the cost of buying the option (or buying a synthetic option, namely the replicating portfolio):

Profit = €10 - €8.61 = €1.39

If left unhedged, FI might end up loosing money. For instance, if the stock price increases to €133.10 in 3-month, the call option would be exercised. As FI is short on the option, it would receive the strike price (€100) but would have to deliver a stock worth €133.10, a loss of €33.10, much greater than the premium received initially.
In order to hedge, FI should by a synthetic call option. Using [15] and [16] the initial value of δ and M for the call option are:
$\delta = (13.43-2.43)/(110-90) = 0.55$
$M = (2.43 \times 1.10 - 13.43 \times 0.9)/[(1.10-0.9)(1.005)] = -46.83$
FI should thus buy 0.55 shares and borrow 46.83. The cost associated with this transaction is:

$$0.55 \times 100 - 46.83 = 8.61$$

This portfolio will provide a hedge for the first period.
But both δ and *M* vary as shown in the next two tables

Dynamic of δ (delta)

Period	0	1	2
			1.000
		0.766	
	0.550		0.449
		0.258	
			0.000

Dynamic of *M*

Period	0	1	2
			-99.50
		-70.81	
	-46.83		-39.85
		-20.82	
			0.00

Suppose, for instance that the stock increases to €110 in period 1.
The market value of the replicating portfolio set up in period 0 is:

$$(0.55)(110) – (46.83)(1.005) = 13.43$$

But as both δ and *M* have changed, the composition of the replicating portfolio should be modified in order to be hedged for the second period.
As δ now equals 0.766 and *M* = -70.81, FI should buy 0.216 (=0.766-0.550) additional shares and increase its borrowing by 23.76 (= 70.81 – 46.83 x 1.005). Note that the additional borrowing is the exact amount required to buy the additional shares. The dynamic hedging strategy is self financing.

Expected return and risk on an option

As the composition of the replicating portfolio changes over time, so does its expected return and its risk.
In the replicating portfolio, the fraction w of the total value invested in the stock is:

$$w = (\delta S)/V \qquad [18]$$

The expected return per period for the option is equal to the expected return for the replicating portfolio. Denote by r_{stock} the expected return per period for the stock and by r_{option} the expected return per period for the option. From portfolio theory, we get:

$$r_{option} = w\, r_{stock} + (1\text{-}w)\, r_f \qquad [19]$$

Even assuming r_{stock} and r_f constant, changing value for w will change the expected return for this option.

This explains why valuing an option using a risk-adjusted discount rate is impossible. As the expected return for the option changes over time, finding the appropriate discount rate is impossible.
By continuing this line of reasoning, we can also gain some insight on the beta of an option. Remember that the beta of a portfolio is the weighted average of the betas of individual security. As the beta of a bond is zero, the beta of an option (equal to the beta of the replicating portfolio) is:

$$\beta_{option} = w\, \beta_{\text{stock}} = (\delta S/V)\, \beta_{\text{stock}} \qquad [20]$$

Example 7: expected return and risk for the 3-month European call option

Let's continue analyzing examples 1, 3 and 5.
The expected return on the stock was calculated in example 1 using the true probabilities as:

$$r_{stock} = (0.6)(+10\%) + (0.4)(-10\%) = 2\%$$

The expected return on the option could be calculated in a similar way. The two possible returns on the call option during period 1 (using the binomial tree for the call option in example 3) are:

$(13.43-8.16)/8.16 = +64.5\%$ if the stock price goes up

$(2.43-8.16)/8.16 = -70.2\%$ if the stock price goes up

Using the real probabilities of ups and downs, the expected return for the call option is:

$$r_{option} = (0.6)(+64.5\%)+(0.4)(-70.2\%) = +10.6\%$$

Another way to obtain the expected return on the call option is to calculate w, the fraction invested in stock in the replicating portfolio. As $\delta=0.55$, the amount invested in the stock is

$$\delta S = (0.55)(€100) = €55.$$

The total value of the replicating portfolio is

$$V = 8.16$$

Hence

$$w = 55/8.16 = 6.738$$

The expected return on the option is:

$$r_{option} = (6.738)(2\%) +(1-6.738)(0.5\%)= + 10.6\%.$$

What about the risk for the option?
Assume that the CAPM holds and the risk premium on the market portfolio is 1% per month (not a very realistic figure but it will do). The expected return for the stock is equal to 2%. According the CAPM:

$$r_{stock} = r_f + (\textit{Market risk premium}) \times \beta_{stock}$$

$$2\% = 0.5\% + (1\%) \times \beta_{stock}$$

Solving for β_{stock}:

$$\beta_{stock} = 1.5$$

As for the call option:

$$\beta_{option} = w\ \beta_{stock} = 6.738 \text{ x } 1.5 = 10.1$$

Is the expected return in line with the risk? According to CAPM:

$$r_{option} = r_f + (\textit{Market risk premium}) \text{ x } \beta_{option}$$

$$= 0.5\% + (1\%)(10.1)$$

$$= 10.6\%$$

Setting r_f, u and d

Up to now, the analysis has proceeded based on fixed values for the time step Δt, the risk-free interest rate r_f and the up and down ratios u and d. We now address the question of how to set these parameters.

In this section we will consider the time step as fixed. How to choose the length of a period will be analyzed in the next section

Consider first the interest rate. Remember that, up to now, the risk-free interest rate was expressed as an interest rate per period. However, interest rates are usually quoted per annum. Moreover, care should be taken to take into account the compounding periodicity.

A widely used convention used in the option literature is expressed the risk-free interest rate as a continuously compounded interest rate per annum. Let us denote as r. The relationship between the interest rate per period r_f and the continuously compounded interest rate per annum is

$$1+r_f = e^{r\Delta t} \qquad [21]$$

As an example, if the risk-less interest with continuous compounding is 6% per annum, the risk-free interest for a time step of one month (Δt = 1/12) is:

$$r_f = e^{0.06/12} - 1 = 0.501\%$$

Consider next the up and down ratios. Their values are function of the volatility of the underlying stock. The volatility for a stock is the standard deviation of annual return. Denote it as σ. The local volatility for a time interval Δt is define as the standard deviation of the return over the period. Its relation to the annual volatility is:

$$\sigma_{loc} = \sigma\sqrt{\Delta t}$$

The rational behind this formula is that *variance* is proportional to the length of the time step:

$$\sigma_{loc}^2 = \sigma^2 \Delta t$$

Suppose, for instance, that σ = 30% and Δt = 1/52 (one step per week). Then:

$$\sigma_{loc} = 0.30\sqrt{\frac{1}{52}}. = 0.0416 = 4.16\%$$

The values for the up and down ratios are chosen to make the local volatility at each step in the binomial tree equal to the local volatility. Up to a very good approximation, this is accomplished by using formulas due to Cox, Ross and Rubinstein (1979- referred to as CRR):

$$u = e^{\sigma\sqrt{\Delta t}}, \qquad d = 1/u \qquad [22]$$

Note that for a small value of Δt:

$$u \cong 1 + \sigma\sqrt{\Delta t} \cong 1 + \sigma_{loc}$$

$$d \cong 1 - \sigma\sqrt{\Delta t} \cong 1 - \sigma_{loc}$$

By setting $d = 1/u$, CRR center the binomial tree around the initial price: an up followed by a down leads back to the initial price.

Example 8: valuing a 6-month call option

Suppose that we want to value a 6-month call option on a non-dividend paying stock. The current price is of the stock is €100 and the strike price is €100. The risk-free interest rate with continuous compounding is 6% per annum. The volatility of the stock is 30% per annum.
We will value this option using a binomial tree with one step per month ($\Delta t = 1/12$).
Using the CRR formulas, the parameters for this tree are:

$$u = e^{0.30\sqrt{1/12}} = 1.0905$$

$$d = 1/1.0905 = 0.9170$$

$$r_f = e^{0.06/12} = 0.05013\%$$

The risk-neutral probability of an stock increase is:

$$p = (1.005013 - 0.9170)/(1.0905 - 0.9170) = 0.50727$$

The following table illustrates the evolution of the stock price. In this presentation, an up movement is on the same line and a down movement is on the next line (this presentation minimizes space in a spreadsheet).

Up →

Down ↓

Period	0	1	2	3	4	5	6
0	100.00	109.05	118.91	129.67	141.40	154.19	168.14
1		91.70	100.00	109.05	118.91	129.67	141.40
2			84.10	91.70	100.00	109.05	118.91
3				77.12	84.10	91.70	100.00
4					70.72	77.12	84.10
5						64.86	70.72
6							59.47

The value of the European call option is obtained in the next table

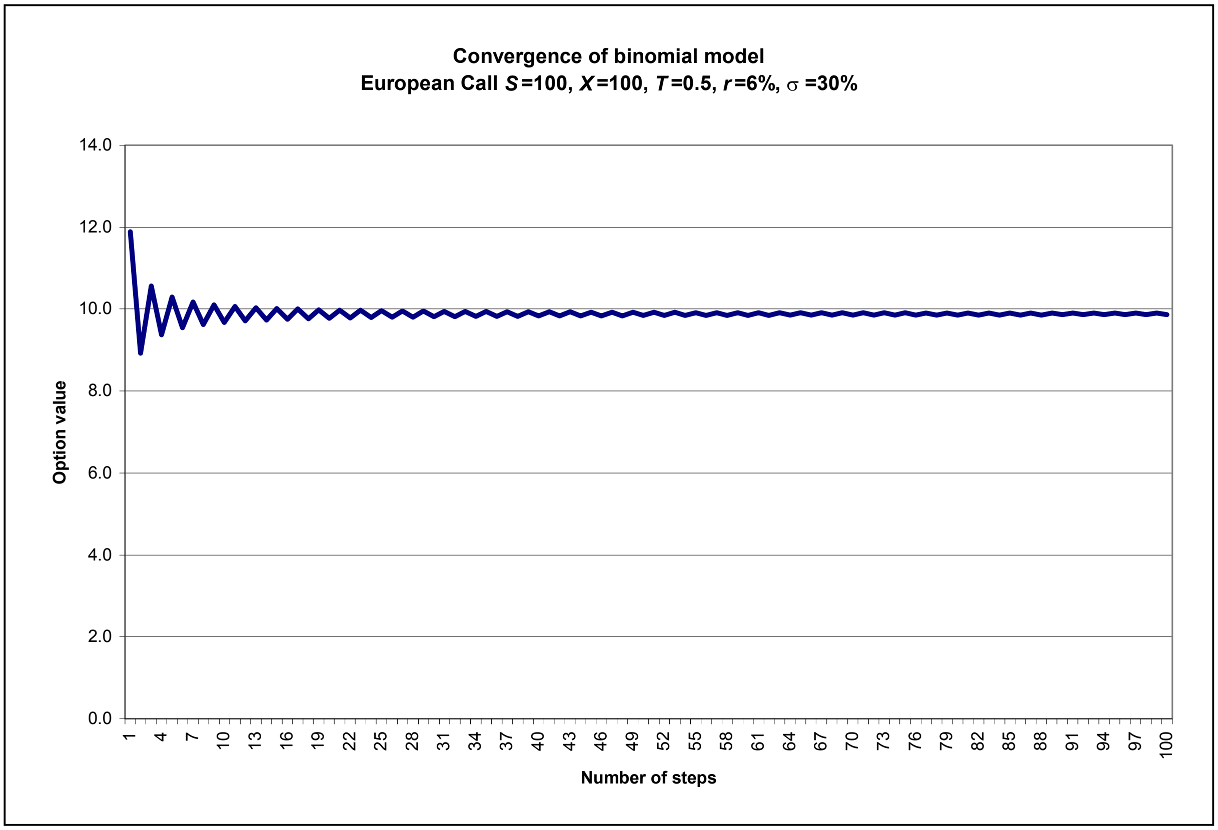
Period	0	1	2	3	4	5	6
0	9.54	14.66	21.81	31.16	42.39	54.69	68.14
1		4.36	7.46	12.41	19.91	30.17	41.40
2			1.23	2.43	4.82	9.55	18.91
3				0.00	0.00	0.00	0.00
4					0.00	0.00	0.00
5						0.00	0.00
6							0.00

Choosing the time step

Let us now analyze the choice of the time step. Remember that the stock price can only take two possible values at the end of a time step. Quite obviously, the longer the time step, the less acceptable is this assumption.
To illustrate, let us value the 8-month call option of example 8 by changing the time step. (The visual basic code used is provided in Appendix 1).

Number of time steps	Value of call option
1	11.89
2	8.93
3	10.56
4	9.38
5	10.29
6	9.54
7	10.17
8	9.62
9	10.11
10	9.68
15	10.02
20	9.78
25	9.96
30	9.81
40	9.83
50	9.84
100	9.86
1000	9.88

As illustrated by the following figure, the call value converges to some value as the number of time steps increases.



For a European option, the option price calculated with the binomial model converges to the value from the Black Scholes formula as the number of time steps increases.
Note the oscillation of the option price. Odd numbers of periods overvalue the option whereas even numbers undervalue the option. In practice, the number of time steps should be 30 or more in order to achieve sufficient accuracy.

References

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